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Note

A three-class association scheme on the flags of a finite projective plane and a (PBIB) design defined by the incidence of the flags and the Baer subplanes in PG(2, q^2)

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0. Summary

First we define relations between the $v=(s^2+s+1)(s+1)$ flags (point-line incident pairs) of a finite projective plane of order s. Two flags $a \equiv (p, l)$ and $b \equiv (p', l')$, where p and p' are two points and l and l' are two lines of the projective plane, are defined to be *first associates* if either $p=p'$ or $l=l'$: *second associates* if $p \neq p'$, $l \neq l'$ but either p is incident also with l' or p' is incident also with l: *third associates*, otherwise.

We show that these relations define a three-class association scheme on $v=(s^2+s+1)(s+1)$ flags with $n_1=2s, n_2=2s^2$ and $n_3=s^3$(n_i denotes the number of i-th associates of a given flag, $i=1,2,3$) and the association matrices are given in Section 2.

If a finite projective plane of order s admits a subplane of order q, then it is known (Bruck [2]) that either $s=q^2$ or $s \geqslant q^2+q$. If $s=q^2$, then the plane of order s has a subplane of order q which is called a *Baer subplane*. In a Desarguesian finite

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projective plane of order q^2, PG$(2, q^2)$, all subplanes are Baer subplanes of order q and there are $b = q^3(q^3+1)(q^2+1)$ Baer subplanes in PG$(2, q^2)$.

Next, we show that the incidence of the flags and the Baer subplanes of PG$(2, q^2)$ defines an incomplete block design (called a partially balanced incomplete block design) with parameter.

$$v = (q^4+q^2+1)(q^2+1), \qquad b = q^3(q^3+1)(q^2+1), \qquad r = q^3(q+1)^2,$$
$$k = (q^2+q+1)(q+1), \qquad \lambda_1 = q^2(q+1)^2, \qquad \lambda_2 = q(q+1)^2, \qquad \lambda_3 = (q+1)^2.$$

based on the three-class association scheme defined on the flags.

1. Introduction, definitions

For a definition of a finite projective plane, please see Dembowski [4].

A point-line pair $(p, l) p \in \mathscr{P}, l \in \mathscr{L}$ such that p is incident with l is called a flag. There are $(s^2+s+1)(s+1)$ flags in the plane.

A *subplane* of a finite projective plane Π is a subset $\mathscr{C}$ of points and lines, which itself is a projective plane.

A closed subset $\mathscr{C}$ of Π is called a *Baer subset* or a Baer subplane in case it is a subplane, if it satisfies the following conditions:

(1) Every point of Π is incident with a line of $\mathscr{C}$.

(2) Every line of Π is incident with a point of $\mathscr{C}$.

Thus a Baer subplane if it exists is a maximal subplane of Π.

Let Π be a projective plane of order s and $\mathscr{C}$ a proper subplane of Π, of order q. Then:

(i) $q^2 = s$ if and only if $\mathscr{C}$ is a Baer subplane, and

(ii) $q^2 + q \leqslant s$ if $\mathscr{C}$ is not a Baer subplane.

If Π is a finite Desarguesian projective plane PG$(2, s)$ of square order that is, $s = q^2$, every subplane of Π is a Baer subplane and every quadrangle can be completed to a unique Baer subplane. The number of Baer subplanes in Π is then, $q^3(q^3+1)(q^2+1)$. The definitions and the results given above can be found in Dembowski [4] and Cofman [3].

For a definition of an association scheme and their properties, please see Bose and Mesner [1] and MacWilliams and Sloane [5, Chapter 21].

Given an m class association scheme with its parameters, a partially balanced incomplete blocks (pbib) design is an arrangement of the v points in b sets (blocks) of k points each such that each point is contained in r sets and a pair of points which are i-th associates occur together in λ_i sets $i = 1, \ldots, m$.

2. A three-class association scheme defined on the flags of a finite projective plane of order s

We define two flags $a = (p, l)$ and $b = (p', l')$ where p, p' are two points and l, l' are two lines of the plane, to be *first associates* if either $p = p'$ or $l = l'$; *second associates* if

$p \neq p', l \neq l'$ but either p is incident also with l' or p' is incident also with l: *third associates*, otherwise. The number of flags in the plane is $v=(s^2+s+1)(s+1)$.

Given a flag $a=(p,l)$, its first associates are the flags of the types (p,l') and (p',l). Since there are s lines other than l, incident with p and s points other than p incident with l, the number of first associates of (p,l) is $n_1=2s$.

The second associates of the flag $a=(p,l)$ are the flags (p',l'), $p \neq p', l \neq l'$ but either p' is incident with l or p is incident with l'. Now there are s points other than p, on l and each such point is incident with s lines other than l. Again there are s lines other than l incident with p and each line is incident with s points other than p. Hence the number of second associates of (p,l) is $2s^2$.

The third associates of the flag $a=(p,l)$ are the flags $(p',l'), p \neq p'$ and $l \neq l'$ and neither p is incident with l' nor p' is incident with l. Since there are s points other than p, incident with l and s lines other than l incident with each one of these points, the number of third associates of $a=(p,l)$ is $n_3=s^3$.

Consider two flags $a=(p,l)$ and $b=(p,l'), l \neq l'$ which are first associates. Then the flags which are first associates of both a and b, are of the type (p,l'') and hence $p_{11}^1(a,b)=s-1$ which is independent of a and b.

Let us calculate $p_{12}^1(a,b)$. It is clear that (p',l) is a first associate of $a=(p,l)$ and a second associate of $b=(p,l')$ and there are s such flags. Hence $p_{12}^1(a,b)=s$, which is independent of a and b.

Suppose now that $a=(p,l)$ and $b=(p',l'), p \neq p', l \neq l'$ but p is incident with l', so that a and b are second associates. Let us calculate, say, $p_{23}^2(a,b)$. A flag (p'',l'') where p'' is incident with l but $p'' \neq p$ is a second associate of $a=(p,l)$ and a third associate of $b=(p',l')$ provided l'' is not the line joining p' and p''. There are $s(s-1)$ such flags. Again, a flag (p^*,l^*) where l^* is one of the lines through p other than l and l', is a second associate of a and a third associate of b. And there are $s(s-1)$ such flags. Hence $p_{23}^2(a,b)=2s(s-1)$. In this manner, we have calculated all the p_{jk}^i's, $i,j,k=1,2,3$, which are given below:

$$P_1=(p_{ij}^1)=\begin{bmatrix} s-1 & s & 0 \\ s & s(s-1) & s^2 \\ 0 & s^2 & (s-1)s^2 \end{bmatrix},$$

$$P_2=(p_{ij}^2)=\begin{bmatrix} 1 & s-1 & s \\ s-1 & s & 2s(s-1) \\ s & 2s(s-1) & s(s-1)^2 \end{bmatrix},$$

$$P_3=(p_{ij}^3)=\begin{bmatrix} 0 & 2 & 2(s-1) \\ 2 & 4(s-1) & 2(s-1)^2 \\ 2(s-1) & 2(s-1)^2 & (s-1)(s^2-s+1) \end{bmatrix}.$$

3. Incidence of flags and Baer subplanes in PG(2, q^2). A PBIB design

There are $q^3(q^3+1)(q^2+1)$ Baer subplanes, each of order q in PG$(2,q^2)$. We also know that each quadrangle can be completed to a unique Baer subplane.

Each Baer subplane consists of $(q^2+q+1)(q+1)$ flags. Now given a flag (p,l) the number of quadrangles in PG$(2,q^2)$, which includes (p,l) is $q^2\cdot q^4(q^2-1)^2=q^6(q^2-1)^2$. But each Baer subplane of order q, which includes (p,l), has $q\cdot q^2(q-1)^2=q^3(q-1)^2$ distinct quadrangles. Hence the number of Baer subplanes incident with a given flag (p,l) is $q^6(q^2-1)^2/q^3(q-1)^2=q^3(q+1)^2=r$ (say).

Now, consider two flags a and b which are first associates, say,

$$a=(p,l) \quad \text{and} \quad b=(p',l), \quad p\neq p'.$$

The number of quadrangles in PG$(2,q^2)$ which includes the flags a and b is $q^4(q^2-1)^2$ and the number of quadrangles in a Baer subplane which includes a and b is $q^2(q-1)^2$. Hence the number of Baer subplanes which are incident with both the flags $a=(p,l)$ and $b=(p',l)$ is $\lambda_1=q^4(q^2-1)^2/q^2(q-1)^2=q^2(q+1)^2$.

Let $a=(p,l)$ and $c=(p',l'), p\neq p', l\neq l'$ be two flags which are second associates and, say, p is incident with l'. Then the number of quadrangles which include both a and c is $q^2(q^2-1)^2$ and the number of quadrangles in a Baer subplane which includes both a and c is $q(q-1)^2$. Hence a and c occur together in $\lambda_2=q^2(q^2-1)^2/q(q-1)^2 \;=q(q+1)^2$.

Now consider two flags $a=(p,l)$ and $d=(p',l'), p\neq p', l\neq l', p$ not incident with l' and p' not incident with l, so that a and d are third associates. Suppose l and l' meet at the point $p''\neq p, p''\neq p'$. Any quadrangle which includes the flags (p,l) and (p',l') must also include p'' the point of intersection of l and l'. Hence the number of quadrangles in PG$(2,s^2)$ which include both a and d is $(q^2-1)^2$ and the number of quadrangles in a Baer subplane which include both a and d is $(q-1)^2$. Hence a and d which are third associates, occur together in $\lambda_3=(q^2-1)^2/(q-1)^2=(q+1)^2$.

Thus the incidence of the flags and the Baer subplanes in PG$(2,q^2)$ defines a partially balanced incomplete block (pbib) design with

$$v=(q^4+q^2+1)(q^2+1), \qquad b=q^3(q^3+1)(q^2+1), \qquad r=q^3(q+1)^2,$$

$$k=(q^2+q+1)(q+1), \qquad \lambda_1=q^2(q+1)^2, \qquad \lambda_2=q(q+1)^2, \qquad \lambda_3=(q+1)^2$$

with the same association matrices as in Section 2, where s is to be replaced by q^2.

References

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